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THE IMPACT OF A MASS ON A BEAM

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Abstract—The problem of the impact of a mass on a beam of infinite extent is first examined within the context of Bernoulli–Euler beam theory and it is shown that this leads to an infinite deceleration of the mass at the moment of impact. This result arises from the fact that the simple beam theory predicts an infinite velocity for waves of vanishingly small wavelength. Thus the use of the Timoshenko beam equations which predict finite wave velocities may be expected to lead to more realistic results. It is shown that a finite value of the initial deceleration is obtained by using these equations. Certain conclusions of interest in the design of beam type highway guardrails are drawn from the results.

INTRODUCTION

THE problem of the impact of a mass on a beam is of interest in partially understanding the behavior of an automobile which collides with a highway guard rail and has, in addition, an intrinsic interest as a problem of structural dynamics. The total response of a beam type highway fence is clearly a complex combination of elastic and plastic behavior, both in the fence and in the impacting vehicle. Nevertheless, it may be conjectured that the initial stages of the impact, which might be expected to produce the highest decelerations on the passengers and during which time energy dissipation due to plastic flow is still of minor importance, might be adequately represented by an elastic approach.

To this end, we illustrate here a Laplace transform method of solution for the impact of a mass on an elastic beam of infinite extent in which primary concern will be with the motion of the mass after impact. It will be shown that the solution on the basis of simple (Bernoulli–Euler) beam theory can be given in closed form, but while it provides physically reasonable values of velocity and displacement of the mass immediately after impact, it leads to an infinite deceleration of the mass at the instant of impact. For the purpose of providing a numerical estimate of the maximum deceleration experienced by passengers such a solution is clearly useless.

It is well known that the Bernoulli–Euler theory of dynamic response of beams predicts that the disturbance due to a suddenly applied load propagates infinitely rapidly. However, we will show that the impact of a mass on a tightly stretched string has a finite value of initial deceleration. It may be conjectured then that the infinite value of the deceleration is a result of the infinite wave velocity of the simple beam theory and that it may be removed by using the Timoshenko beam theory in which finite wave velocities occur. As will be shown, the Timoshenko beam equations do, in fact, predict a finite value of initial deceleration.

IMPACT ON BERNOULLI–EULER BEAM

The equation of displacement, $w(x, t)$ of a beam of cross sectional area A, moment of inertia I, density ρ and modulus of elasticity E is, on the basis of Bernoulli–Euler theory,

$$EIw,_{xxxx} + A\rho w,_{tt} = -p(x, t) \tag{1}$$

where $p(x, t)$ is the external applied force on the beam which in this case is given by

$$p(x, t) = -Mw,_{tt}|_{x=0}\delta(x) \tag{2}$$

M being the mass (impacting at $t = 0$ at $x = 0$) and $\delta(x)$ the Dirac delta function.

Application of the Laplace transform to equation (1) with equation (2) gives

$$EI\bar{w},_{xxxx} + A\rho s^2\bar{w} = -M(s^2\bar{w}(0, s) - V_0)\delta(x) \tag{3}$$

where s is the transform parameter and bars above a symbol denote its transform, and V_0 is the impact velocity of the mass. From beam on elastic foundation theory [1], we know the solution to the transformed equation is

$$\bar{w}(x, s) = -M(s^2\bar{w}(0, s) - V_0)\cdot\frac{1}{8EIk^3s^{\frac{3}{4}}}e^{-ks^{\frac{1}{4}}|x|}.(\cos ks^{\frac{1}{4}}|x| + \sin ks^{\frac{1}{4}}|x|); \qquad k = \left(\frac{\rho A}{4EI}\right)^{\frac{1}{4}}.$$

The transform $\bar{w}(0, s)$ is thus

$$\bar{w}(0, s) = V_0/s^{\frac{3}{4}}(s^{\frac{1}{4}} + \alpha); \qquad \alpha = 8EIk^3/M,$$

which may be inverted by standard methods [2] to give

$$\begin{aligned} w(0, t) &= V_0\left\{\int_0^t e^{\alpha^2\tau}\operatorname{erfc}(\alpha\sqrt{\tau})\,d\tau\right\} \\ w,_t(0, t) &= V_0\{e^{\alpha^2 t}\operatorname{erfc}(\alpha\sqrt{t})\} \\ w,_{tt}(0, t) &= V_0\{\alpha^2 e^{\alpha^2 t}\operatorname{erfc}(\alpha\sqrt{t}) - \tfrac{1}{4}\sqrt{(\alpha^2/t)}\}. \end{aligned} \tag{4}$$

It is clear from the last of these results that this solution, while providing reasonable velocity and displacement solutions indicates an infinite deceleration at the instant of impact. Intuitively, the reason for this result is that the slope of the beam under the point of application of the load must be continuous and thus at the moment of impact the mass engages a finite portion of the beam, this being intimately connected with the infinite wave velocity. If we look at the same problem for a string of linear density ρ and tension T in which case the equation for displacement $w(x, t)$ is

$$Tw,_{xx} - \rho w,_{tt} = Mw,_{tt}|_{x=0}\delta(x) \tag{5}$$

we obtain, on the basis of the argument deriving equation (4) from equation (1) and equation (2), the solutions

$$\begin{aligned} w(0, t) &= V_0(Mc/2T)(1 - e^{-(2T/Mc)t}) \\ w,_t(0, t) &= V_0\, e^{-(2T/Mc)t} \\ w,_{tt}(0, t) &= -V_0(2T/Mc)\, e^{-(2T/Mc)t} \end{aligned} \tag{6}$$

This shows that where a finite wave velocity exists, a finite deceleration is obtained. In fact, in this case it is clear that at the instant of impact the portion of the string carried forward by the mass has zero mass.

It is interesting to obtain the bending moment under the point of impact. Writing the equation for $\bar{w}(x, s)$ in the form

$$\bar{w}(x, s) = \bar{w}(0, s)\, e^{-ks^{\frac{1}{4}}|x|}(\cos ks^{\frac{1}{4}}|x| + \sin ks^{\frac{1}{4}}|x|)$$

and differentiating twice with respect to x gives for the bending moment, $N(x, t)$, the transform

$$\bar{N}(x, s) = 2EIk^2\bar{w}(0, s)s\, e^{-ks^{\frac{1}{4}}|x|}(\cos ks^{\frac{1}{4}}|x| - \sin ks^{\frac{1}{4}}|x|)$$

from which we obtain

$$N(0, t) = 2EIk^2 V_0\, e^{\alpha^2 t}\, \mathrm{erfc}\,(\alpha\sqrt{t}).$$

The result for the bending moment is a multiple of that for the velocity under the load and the formula may be taken as a physically reasonable one. The initial value of the moment, $2EIk^2V_0$, is curiously enough independent of the magnitude of the impacting mass. Since the moment decays from this initial value (for a finite impacting mass) for the impact to be elastic it is enough that $2EIk^2V_0 \leq N_y$, the yield moment of the beam cross section. Since $N_y = I\sigma_y/y$ where σ_y is the yield stress and y the distance from the centroidal axis of the beam to the extreme fiber, the limiting velocity for an elastic impact is given by

$$V_0/c < (r/y)\epsilon_y$$

where $r = (I/A)^{\frac{1}{2}}$ is the radius of gyration of the section and $\epsilon_y = \sigma_y/E$ is the yield strain and $c = (E/\rho)^{\frac{1}{2}}$ the velocity of longitudinal waves in the beam material.

IMPACT ON TIMOSHENKO BEAM

On the basis of the foregoing analysis, it is clear that a useful estimate of the initial deceleration can only be obtained if wave propagation effects in the beam are included. Thus we consider the impact of a mass on a Timoshenko beam. The equation of the displacement is given, for example, in Flügge and Zajac [3] and takes the form

$$w_{,xxxx} - \left(\frac{1}{c_1^2} + \frac{1}{c_2^2}\right)w_{,xxtt} + \frac{1}{c_1^2}\cdot\frac{1}{c_2^2}w_{,tttt}$$
$$+ (\rho A_i/EI_e)w_{,tt} = (1/EI_e)\left[p - (EI_e/GA_s)\left(p_{,xx} - \frac{1}{c_1^2}p_{,tt}\right)\right] \qquad (7)$$

where $c_1^2 = EI_e/\rho I_i$ and $c_2^2 = GA_s/\rho A_i$ and G is the shear modulus, I_e is the elastic moment of inertia, I_i the mass moment of inertia, A_s the effective shear area and A_i the mass per unit length. We note that I_e need not be identical with I_i nor A_s with A_i.

Substituting $p = P(t)\delta(x)$ and taking the Laplace transform gives

$$\bar{w}_{,xxxx} - (s^2/c_1^2 + s^2/c_2^2)\bar{w}_{,xx} + (s^4/c_1^2c_2^2 + s^2\rho A_i/EI_e)\bar{w}$$
$$= \bar{P}(s)\{\delta(x) - EI_e(\delta''(x) - s^2/c_2^2\delta(x))/GA_s\}/EI_e \qquad (8)$$

The solution of (8) for $x = 0$ is

$$\bar{w}(x, s) = A\, e^{\lambda_1 x} + B\, e^{\lambda_2 x}; \qquad x < 0,$$
$$\bar{w}(x, s) = C\, e^{\lambda_3 x} + D\, e^{\lambda_4 x}; \qquad x > 0,$$

where

$$\lambda_1 = \frac{s}{\sqrt{2}}\left\{\left(\frac{1}{c_1^2} + \frac{1}{c_2^2}\right) + \left[\left(\frac{1}{c_1^2} - \frac{1}{c_2^2}\right)^2 - \frac{4\rho A_i}{EI_e}\cdot\frac{1}{s^2}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$\lambda_2 = \frac{s}{\sqrt{2}}\left\{\left(\frac{1}{c_1^2} + \frac{1}{c_2^2}\right) - \left[\left(\frac{1}{c_1^2} - \frac{1}{c_2^2}\right)^2 - \frac{4\rho A_i}{EI_e}\cdot\frac{1}{s^2}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$\lambda_3 = -\frac{s}{\sqrt{2}}\left\{\left(\frac{1}{c_1^2} + \frac{1}{c_2^2}\right) + \left[\left(\frac{1}{c_1^2} - \frac{1}{c_2^2}\right)^2 - \frac{4\rho A_i}{EI_e}\cdot\frac{1}{s^2}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$\lambda_4 = -\frac{s}{\sqrt{2}}\left\{\left(\frac{1}{c_1^2} + \frac{1}{c_2^2}\right) - \left[\left(\frac{1}{c_1^2} - \frac{1}{c_2^2}\right)^2 - \frac{4\rho A_i}{EI_e}\cdot\frac{1}{s^2}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

The values of A, B, C, D may be obtained by integration of equation (8) across the load point $x = 0$. We recall that

$$\int_{-L}^{L} \delta^{(r)}(x) f(x)\,\mathrm{d}x = (-1)^r r!\left.\frac{\mathrm{d}^r f}{\mathrm{d}x^r}\right|_{x=0}; \qquad L > 0.$$

This result follows formally by integration by parts r times. For a discussion of the validity of this procedure for generalized functions see, for example, Friedman [4]. Thus multiplication of each side of equation (8) by x^r, $r = 0, 1, 2, 3$, and integration of each side from $-\epsilon$ to $+\epsilon$, $\epsilon > 0$, gives in the limit as $\epsilon \to 0$

$$[\bar{w}'''] - s^2\left(\frac{1}{c_1^2} + \frac{1}{c_2^2}\right)[\bar{w}'] = \frac{\bar{P}(s)}{EI_e}\left(1 + \frac{\rho I_i}{GA_s}s^2\right)$$

$$[\bar{w}''] = 0$$

$$[\bar{w}'] = -\frac{\bar{P}(s)}{GA}$$

$$[\bar{w}] = 0$$

where

$$[f] = \lim_{\epsilon \to 0}(f(+\epsilon) - f(-\epsilon)); \qquad \epsilon > 0.$$

This leads to four matching conditions at $x = 0$ from which are obtained the four constants A, B, C, D. The set of equations which results is a special case of the set

$$\sum_{j}^{4} (\lambda_j)^{i-1} x_j = y_i, \qquad i = 1 \text{ to } 4,$$

and the solution is

$$x_1 = \frac{1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)}\{\lambda_2\lambda_3\lambda_4 y_1 - (\lambda_2\lambda_3 + \lambda_3\lambda_4 + \lambda_4\lambda_2)y_2 + (\lambda_2 + \lambda_3 + \lambda_4)y_3 - y_4\}$$

the solution for x_i being obtained by interchanging the indices i and 1 in the above. The result of interest here is $\bar{w}(0, s) = A + B$ which takes the form

$$\bar{w}(0, s) = \frac{(\lambda_2\lambda_3 + \lambda_3\lambda_4 + \lambda_4\lambda_2 + s^2/c_2^2)[\bar{P}(s)/GA_s - \bar{P}(s)/EI_e]}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)}$$

$$+ \frac{(\lambda_3\lambda_4 + \lambda_4\lambda_1 + \lambda_1\lambda_3 + s^2/c_2^2)[\bar{P}(s)/GA_s - \bar{P}(s)/EI_e]}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)(\lambda_2 - \lambda_1)}$$

$$= \bar{P}(s)\Omega(s), \text{ say.}$$

Now

$$\bar{P}(s) = -M(s^2\bar{w}(0, s) - V_0)$$

from which it follows that

$$\bar{w}(0, s) = V_0/(s^2 + 1/M\Omega(s)). \tag{9}$$

To obtain the initial deceleration we only need obtain the assymptotic behavior of this transform for large s. It is, however, not possible to obtain the result from the formula

$$f(0) = \lim_{s\to\infty} s\bar{f}(s)$$

since in this and the other cases we obtain an infinite result by the initial condition that $w(0) = 0$; $w,_t(0) = V_0$; this is of course the acceleration experienced by the point $x = 0$ of the beam not the deceleration experienced by the mass. We obtain the required result by noting that

$$\Omega(s) = c_2/2GA_s s + \text{terms in } 1/s^2 \text{ etc.}$$

and thus expanding equation (9) in a series in $1/s$ leads to

$$s^2\bar{w}(0, s) = V_0\left\{1 - \frac{2GA_s}{Mc_2 s} + \text{terms in } 1/s^2, \text{etc.}\right\}$$

and

$$w,_{tt}(0, 0) = -\frac{2V_0GA_s}{Mc_2}$$

CONCLUSION

The results obtained for the deceleration of the impacting mass and beam bending moment suggest certain conclusions relevant to the design of beam type highway guard rails. From the formula for the moment it is clear that the ratio r/y should be made as high as possible to maximize the range of impacting velocities which produce elastic impacts. This suggests the use of wide flange beams on the side as a suitable design. For such beams this ratio may be as high as 0·90 or higher. For a steel beam the velocity at which the impact becomes plastic may be around 14 m.p.h. This is a significant result since the statistics for vehicle accidents with fixed objects in California in 1963 indicate that 80 per cent of all accidents occur at velocities less than 40 m.p.h. at angles of impact less than 20°. However, 40 m.p.h. at 20° represents a normal component of 14 m.p.h. It seems possible then, that by suitable design the damage to the guard rail could be minimized over a wide range of accidents. The use of wide flange beams is also suggested by the result for the initial deceleration of the impacting vehicle. The value of this deceleration may be minimized by providing a small shear area A_s, and this is characteristic of wide flange beams. Another design solution in which the deceleration may be minimized while maintaining a large r/y ratio might be a guardrail of two steel plates separated by a timber core or merely by timber spacers. It is, of course, possible that the results obtained here might suggest other types of design and it is hoped that the results might prove useful in the practical design of guard rails.

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Résumé—Le problème du choc d'une masse sur une poutre infinie est d'abord examiné à la lumière de la Théorie de Bernoulli-Euler sur les poutres, et l'on constate que ceci conduit à un rallentissement infini de la masse au moment du choc. Ce résultat provient du fait que la théorie des poutres simples prédit une vélocité infinie des ondes de petites longueurs d'ondes de disparition. Par conséquent, l'on pourrait s'attendre à ce que l'emploi des équations de Timoshenko qui prédisent des vélocités d'ondes finies, conduisent à des résultats plus réalistes. L'on démontre qu'une valeur finie du rallentissement initial peut être obtenue à l'aide de ces équations. Quelques conclusions intéressantes concernant les modèles des gardefous pour routes et ponts, du genre à poutre, y sont tirées de ces résultats.

Zusammenfassung—Der Stoß einer Masse auf einen Balken von unbegrenztem Ausmaß wird vorerst nach der Bernoulli–Euler Theorie untersucht und es wird gezeigt, daß die Masse im Augenblick des Stoßes unendlich verlangsamt wird. Dies folgt aus der Tatsache, daß die Theorie einfacher Balken eine unendliche Geschwindigkeit für verschwindend kleine Wellenlängen voraussagt. Verwendet man Timoschenkos Gleichungen, die begrenzte Wellengeschwindigkeiten voraussagen so erhält man realere Resultate. Es wird gezeigt, daß mit diesen Gleichungen für die Anfangsverlangsamung begrenzte Werte erhalten werden. Gewisse Schlüsse, die für den Entwurf von Strassengeländern interessant sind werden erwähnt.

Абстракт—Прежде всего исследуется проблема динамического воздействия удара массы на балку бесконечного протяжения в контексте теории балки Бернулли-Эйлера (Bernoulli–Euler), и указывается, что это приводит к бесконечному замедлению массы в момент столкновения-удара. Этот результат возникает из факта, что теория простой балки предсказывает бесконечную скорость для волн исчезающе малой длины волны. Отсюда—применение уравнений балки Тимошенко, которые предсказывают, что конечные скорости волны могут привести к более реальным результатам. Указывается, что при применении этих уравнений получается конечное значение первоначального замедления. Из этих результатов выведены некоторые интересные заключения в проекте балочного типа ограждений дорог общего пользования.